International Journal of Economics, Commerce and Management
United Kingdom ISSN 2348 0386 Vol. IX, Issue 8, Aug 2021

http://ijecm.co.uk/

SUPERSYMMETRIC STOCHASTIC DYNAMICS & SELF-ORGANIZED CRITICALITY: AN ECONOMIC DEBATE

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Abstract

Economies are influenced by both random and predetermined factors. The Supersymmetric Theory of Stochastic Dynamics (STSD) is a body of stochastic partial differential equations (PDE) which applies to dynamical systems; economies operate as dynamical systems and thus can be analyzed via this framework. Self-organized criticality (SOC)—the observation that large interactive systems evolve to a critical state whereby a trivial event triggers a sequence of events that impact a variety of elements in the system—attempts to apply concerns to economic systems that are better explained by STSD. This paper defined the two notions and how they juxtapose one another in their application to economic phenomena. The research framework employed both qualitative and quantitative methodologies and shows that STSD more clearly aligns and accomplishes the goals of the broad research program that is Macroeconomics. A major implication of the study is that STSD provides a more apt framework in its solutions to both present and future pertinent Macroeconomic questions, especially in conjunction with the developing field of Fractals.

Keywords: Criticality, Dynamical, Systems, Stochastics, Supersymmetric

INTRODUCTION

When a statistically significant economic event occurs—be it an endogenous shock such as the demonetization in India of 2016 or an exogenous shock such as World War II—the extemporized reaction is often to assume a proportionate magnitude of the causal factor(s). Characteristic of the Enlightenment era was to employ the analytic method in the study of such systems, namely, to decompose the system and subsequently analyze its components (Phillips, 1976). Furthered by criticism such as that of Bertrand Russel in the Analytic School of Philosophy, Organicism argued the converse, i.e., the whole is determinative of its parts. Organicism utilized the Doctrine of Internal Relations (Moore, 1922), which postulates that any entity's relationships constitute a determinative characteristic of that entity. However, that a relationship between entities may grant them characteristics they might not otherwise have does not mean they are definitive. Just as in Darwinism where many characteristics and traits are selected for but are neutral to the success of that species, the fact that a market selects for certain characteristics and traits does not mean they define that market. Similar is the deficiency in applied logic to the converse; broadly, that a prediction can be made about large interactive systems purveying for example the Financial Crisis of 2008 or the Argentine Debt Crisis circa 2000 on the basis of individual components.

Self-organized criticality (SOC) attempts to ameliorate the above inadequate analysis. SOC describes the frequent observation that large interactive systems inherently evolve to a critical state whereby a trivial event triggers a causal sequence of events which in turn impact a variety of elements in the system (Bak, Tang, & Wiesenfield, 1987). The common illustration in the literature is the development of a mountain of sand on a surface. As the simulation, $f(x)$, progresses, the particles begin to pile and create small asymmetric avalanches and reach a critical point, $f(c)$, where:

$$f'(c) = 0 \ \cup \ f'(c) \nexists$$

When a further particle of sand is added there exists the possibility that it will cause a macro or micro event, though the latter is more probable. Correspondingly, the sand mountain illustration is given by the following:

$$p \in U \subseteq V$$

where; X is a topological space and p is a point in X, a neighbourhood of $p \subset V \, of \, X$ with open set U containing p. Effectively, the probabilities of a local avalanche do not approximate a global one.

The studies of Ott et. al (1994) devised similar experiments. If water is poured at a point where two bodies of water bound a piece of land, one can predict its trajectory taking into account geometric structures including geographic ellipses. One can construct numerous fractal properties of the course such that both the point of origin at which the water is poured and its final resting place is unknown by perusing a quadratic map:

$$z_{n+1} = z_n^2 + c$$

Where; the Mandelbrot Set for values of c in the z-plane with critical point $z = 0$ for $X \to X$. This does not follow a chaos theoretic path such as the following:

$$x_{n+1} = \mu x_n(1 - x_n)$$

The tacit assumptions of the sand pile differ insofar that its summation is preconditioned by differentiable manifolds. That is: (U, φ) is differentiable with U as an open set $\in M \in p$ $\wedge \varphi : U \rightarrow \mathbb{R}^n$ forms the defining map, then $f$is differentiable at:
$p \leftrightarrow f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

The structure can be conformed in such a way that observing a local point can help one determine the probability of some future local event; in the case of the sand pile an avalanche, and in the case of the water spillage what path the water will take. However, the experiments differ because the use of fractals excludes the presence of differentiable manifolds. The sand example that is meant to support SOC trends towards negentropy (Brillouin, 1953):

$$\left(-\int_x f(x) \log f(x)\, dx\right) - \left(-\int p_x(u)\, log\, p_x(u)\, du\right)$$

whereas the water experiment properly abides by Newton's Second Law of Thermodynamics—that entropy trends towards greater disorder:

$$S = k_\mathrm{B} \ln \Omega$$

In the experiments conducted by Ott et. al, by imputing the map and object with stochastic and fractal properties it is impossible to predict both the point of origin and the destination; this contrasts with SOC's presumption of an eventual structure formation. To pile the sand, one must assume a semblance of particles of sand to a degree that they eventually do form a pile. The fractal setup rather is given by the following:

$$P = LB + \varepsilon\,(UB - LB)$$

where; P is the water randomly generated; UB & LB are upper and lower bounds, respectively; and ε represents noise. Proponents of SOC impart a cursory treatment of this and conclude that such systems evolve naturally and attune themselves to some critical point, which the experiments conducted by Ott et. al cast doubt on to the extent of evolutionary economic phenomena.

FRAMEWORK

The above discrepancy can be elucidated via stochastic differential equations (SDEs) in the Supersymmetric Theory of Stochastic Dynamics (STSD). STSD formalizes SDE's in a manner that satisfy topological supersymmetry (Doob, 1990). Slavík (2013) conceptualizes SDEs in phase space $X = \mathbb{R}^N$ using a gradient flow vector field with Gaussian noise:

$$x'(t) = -\partial U(x(t)) + (2\Theta)^{1/2}\xi(t)$$

let; $x \in X, \xi \in \mathbb{R}^n$ be the noise variable; where Θ is noise intensity with $\partial U(x)$ at position

$\delta^{ij}\partial_j U(x) \wedge \partial_i U(x)$ for the gradient flow vector; where $U(x)$ is the energy of the dissipative stochastic dynamical system. The SDE denotes a displacement function $x(t) \in X$ for any noise arrangement ξ with an origin Δ function $x(t') = x' \in X$, where $M_{tt'} : \mathrm{X} \longrightarrow \mathrm{X}$ for $x(t) = M_{tt'}(x')$. Independent of whether the noise configurations are differentiable, the displacement function is differentiable with respect to the first derivative function. SDEs establish an indexed set of noise-arrangement-dependent diffeomorphisms for iterative functions of $X = \mathbb{R}^N$ with a collection for all noise-arrangement-reliant diffeomorphisms, respectively. The collection distributes linear pullbacks ϕ, $M^*_{t't}$: $\Omega(X) \longrightarrow \Omega(X)$; where $\Omega(X)$ are wavefunctions, allowing for averaging $M^*_{t't}$ over ξ, corresponding to the Stochastic Evolutionary Operator (SEO), of which the Lefschetz-Hopf theorem may be viewed as the infimum/supremum tensor contraction (Lefschetz, 1926). For the iterative function $f\colon X \longrightarrow X$ as a continuous function with compact triangulation for f having finite fixed points:

$$\sum_{x\in\mathrm{Fix}(f)} i(f,x) = \sum_{x\in\mathrm{Fix}M_{tt'}} \operatorname{sign}\det(\,\delta^i_j - \partial M^i_{tt'}(x)/\partial x^j)$$

let; $\mathrm{Fix}(f)$ denote the set of fixed points of (f) and $i(f,x)$ equate to the index of the fixed point x. If the state is nontrivial, then $\exists\, x \in X : f(x) = x$. The Kronecker tensor is given by δ^i_j with covariant and contravariant index i and j:

$$\delta^i_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

It follows from this that any dynamical system, including the sand pile and fractal landmasses with water, is characterized by stochastic properties (though as illustrated prior is not necessarily entity defining.) In different terms, any proximate deterministic state in a system will always be a function of distal stochasticity. The state of a system can be defined in an array of ways depending upon the degree to which it is ordered, symmetric, or chaotic (Ovchinnikov, 2016). If $X = \mathbb{R}^N$ is ordered, symmetric and integrable, then the system is not in a dissipative state. A chaos theoretic map in $X = \mathbb{R}^N$ is not integrable but ordered. This defines neither program, despite fractals well describing such a process (Baranger). SOC is returned when $X = \mathbb{R}^N$ is ordered and integrable but exhibits topological supersymmetry breaking in its distribution approaching $f'(c) = 0 \cup f'(c) \nexists$. Topological supersymmetry (TS) is the phenomenon wherein the course of time evolution, an infinite number of bundled points converge; when broken by noise it exhibits SOC. Distributions such as the sand pile example exhibit power laws but do not lead to the inference that the systems converge and presuppose some critical point. SOC properly viewed through $X = \mathbb{R}^N$ being not integrable but ordered is merely consistent with spontaneous TS breaking from upgradient, cross-functional noise, as opposed to a point dependent explanation that requires disequilibrium at local vs. global points. The latter serve to optimize for the distribution which is not necessary to explain extrema including saddle points and maxima/minima designed to solve for the distribution in the experiment. Some of this misunderstanding is resultant of the too frequently misunderstood concept of kurtosis, given by:

$$\frac{\mu_4}{\sigma^4} = \frac{\mu_4}{\sigma^4} \geq \left(\frac{\mu_3}{\sigma^3}\right)^2 + 1$$

where; μ_4 is the central moment and σ is the standard deviation of the distribution. Kurtosis does not measure peakedness but only the extremity of the tails in a distribution, or outliers. By predetermining the structure of the outcome in the sand experiment that supports SOC, they overlook the many instances where samples cannot deliver a variance (this could be contrived by additive, non-linear permutating of sand granules not unlike the fractal water experiment.) While this helps to illustrate the reaction delta function reflected in the system at different scales, this leads to an overemphasis at the dissipative/disequilibrium fixed point level such that in its absence it cannot explain results that deviate from power laws, such as could be illustrated in a test in the water experiment freighted with fractal properties.

PRACTICAL IMPLICATIONS

Although the premise of SOC is invalid— that attenuation of critical points are dispositive on the recognition of certain power laws displayed by stochastic dynamical systems— its standpoint regarding the dynamics in $X = \mathbb{R}^N$ agree with STSD. Consequently, the concept of an event that regresses to the mean, or comports with the peakedeness of a distribution forms a different context. From a set of observations there exists a plethora of critical points that could fit the events *ex post*—the corollary being that parameterizion of power laws is to introduce error in regard to kurtosis. In effect, anomalous critical points might be isomorphic though not wholly reflective of anomalous events, which defeats the utility of nesting the former as a linchpin in an explanatory economic framework. How does this acknowledgement apply to Economics?

One conservative takeaway is that if critical points are not independently useful for predicting events, and something more trivial like topological symmetry breaking as opposed to chaos is determinative of some future event that characterizes a future state of a system, it stands to reason that redundancies play a more important role than are appreciated. One piquant example of this details the Hong Kong Monetary Authority (HKMA) which utilized a currency board, following a steady-state of infinite regress policy controls. For example, an expansion of money supply was followed by a decrease in rates, which led to downward pressure on capital inflows which contracted the money supply, thereby causing interest rates to increase, etc. (Radelet & Sachs, 1998). When many other Asian Countries began to experience volatility due to their leveraged central banks which was commonplace to the region in the 1990's, Hong Kong was able to avoid it due to the redundancies built into the system. The central bank was not concerned with optimizing; rather, the system had built in redundancy. To analogize to the sand pile experiment, by the time the pile has accumulated, it is true as SOC predicts that each marginal sand particle serves as a poor predictor of a catastrophic event, even though it may be elucidatory of a local avalanche (which optimization would reflect.) The self-similarity in fractals is more reflective of this iterative process then the contingent one of critical point analysis, where characteristics come to be presumed definitive and elide optimization. To illustrate, both miniscule and enormous payouts are consistent with being kurtotic, though the calculus in decision tree optimization for a particular outcome may be unlikely to price in an accurate prediction. The premise of SOC partly ignores this by granting primacy to critical points, which lends itself to optimization in lieu of redundancy analysis. In the case of a stochastic process, one can apply the Shannon entropy (1948) to a stochastic process of redundancy such that:

$$r = \lim_{n\to\infty} \left(\frac{1}{n}\right) H(M_1, M_2, \dots M_n)$$

where; $H(M)$ denotes the entropy rate of a stochastic process. Though often used to establish and ergo root out redundancy, this can be utilized to increase redundancy at the expense of an overuse of optimization while monitoring waste.

Another related takeaway concerns the event time-horizon—namely, phenomena in the interval of kurtosis—world wars, COVID, etc.—and the emergence therefrom of any ergodicity apropos stochastic, self-organized, self-similar or fractal properties. If the exogenous or endogenous shocks themselves have not achieved ergodicity whether it be due to a lack of impetus, the state of the system, or other factors, it can be difficult to predict their future trajectory let alone the totality or change in their impact. To define:

$$T : X \to X \in (X, \Sigma, \mu) \wedge \mu = 1 \to \forall\, E \in \Sigma\ \exists\, T^{-1}(E) = E :\ \mu(E) = 0\ \vee\ \mu(E) = 1$$

where; $T : X \to X$ is a measure-preserving dynamical system; let $(X, \Sigma, \mu) \wedge \mu = 1$ be a measure space; where T is ergodic. This is well illustrated in the fractal water experiments and perfectly consistent with entropy and Brownian motion, where adjacent objects can experience widely different outcomes. This is also apt consideration for evaluating debt crises such as the Argentine Debt Crisis, the 1997 Asian Financial Crisis, and the Financial Crisis of 2008; importantly, that an increase in the magnitude of a random variable need not reflect those variables change in conditional expectation.

This underscores the notion of cokurtosis, for example per variables X and Y:

$$K(X,X,X,Y) = \frac{E[(X - E[X])^3(Y - E[Y])]}{\sigma_x^3 \sigma_Y}$$

$$K(X,X,Y,Y) = \frac{E[(X - E[X])^2(Y - E[Y])^2]}{\sigma_x^2 \sigma_Y^2}$$

$$K(X,Y,Y,Y) = \frac{E[(X - E[X])\,(Y - E[Y])^3]}{\sigma_x \sigma_Y^3}$$

Thus, optimizing for example an equal probability distribution of one high-yield mortgage default vs. an entire tranche of them clearly does not price in the magnitude of its effects despite the conditional expectation remaining the same. *Mutatis Mutandis,* the default probability of a bond whose duration is shorter than the frequency of an exogenous event whose frequency is rarer than the life of a bond necessitates a significantly larger sample size to draw from to deduct any inference concerning its probability. Therefore, the economic systems lack sufficient redundancy mechanism to cope when exogenous or endogenous events do occur.

The following redundancy formula does not therefore apply to the framework discussed in this article:

$$p = \prod_{i=1}^{n} p_i$$

where; n denotes components; where p_i is the probability of component i failing, and p is equal to system failure. This is because in the sand pile example, each marginal grain is not orthogonal to either local or global extrema.

Ipso facto, framing the problem around failure as SOC does is quite estranged from the fractal thesis evinced in the water experiment, despite the latter being more approximative of physical phenomena such as entropy. Again, this appears to be a symptom of the desire for optimizing rather than granting primacy to the redundancy attendant to self-similarity and shows how the upshot of SOC is more amenable to a fractal framework that concerns itself with fractal power laws that is justified by STSD.

CONCLUSION

Self-Organized Criticality and the Supersymmetric Theory of Stochastic Dynamics have similar aims; however, the former relies upon unrequired premises revealed upon closer inspection of experiments utilized in support of the program and those similar programs which better highlight points of concern. STSD aligns closer with a fractal definition of economic events contained in the framework section which shares a wider range of applicability. Future research needs to be conducted concerning properties of fractals and especially how they interface with topological symmetry and how it applies more specifically to economic phenomena. Moreover, as opposed to focusing on critical phenomena which are not dispositive in their explanation of local or global extrema scholars should attempt to determine the relationship more closely between TS breaking in STSD and trivial and non-trivial events, including global economic crises.

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