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DERIVING A CUBIC TOTAL COST FUNCTION FROM A CUBIC TOTAL COST CURVE

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Abstract

The standard treatment of short run cost curves in managerial economics and intermediate microeconomics classes starts with a cubic total cost function, $TC(Q) = a + bQ + cQ^2 + dQ^3$ and derives the various per-unit cost functions. These functions are then displayed on two graphs – one depicts TC (as well as its components, variable and fixed cost) in (Q, \$) space. The other depicts marginal cost, MC, average variable cost, AVC, and average total cost, ATC in (Q, \$/Q) space. This article presents a series of linked problems that move the student in the opposite direction. Students are asked: Given the graph of a cubic TC curve, can one geometrically derive estimates of coefficients a, b, c, and d on which this curve is based? The answer is yes. In the process, students learn in a more intensive fashion than simply remembering the functions themselves, the difference between per-unit cost and total cost. The ultimate goal is to provide students with a more grounded understanding of the relationships that exist between the various short run cost curves and why the curves hang together as they do. Two final questions link this cost information to competitive market analysis.

Keywords: Cost curve, Cubic total cost function, Microeconomics.

Download Excel file with separate student and faculty sheets

INTRODUCTION

The standard discussion of cost in intermediate microeconomics and managerial economics texts proceeds from tabular information to graphical and from algebraic information to graphical but it does not proceed from graphical information to algebraic. This article presents a series of linked problems that allow students to work through the analysis of moving from the graphical to the algebraic. The value of this exercise is that it solidifies the relationships that exist between the various per-unit cost functions in a much more concrete fashion than by simply providing the algebraic relations on their own. This exercise should be undertaken after students have worked with the total and per-unit cost functions.

To accomplish this task, one needs to provide students with a quick reminder of three geometric concepts they may have learned much earlier in school but have long since forgotten.

One has to do with the geometric interpretation of average as slope. A second has to do with remembering the equation for the minimum of a parabola. The third is to simply note that if you have the graph of TC, then you do not need to draw a VC curve to obtain VC information if you simply consider the FC point to be the origin of a “shifted” Q axis.

The rest of this article presents material with an intermediate level audience (without the benefit of calculus) in mind. The paper begins with a quick summarization of the algebra of cubic cost. This material may be covered in the student’s text but is provided here because texts vary regarding how the cubic equation is presented. In particular, some texts place a minus sign in front of the quadratic coefficient, others force a + sign and then says that the coefficient must be negative (Keat, Young, & and Erfle, 2013, p. 259; Pindyck & Rubinfeld, 2013, p. 249). Other texts show cubic cost curves (or at least cost curves that could be modeled as cubic) but do not discuss the algebra behind those curves (Bernheim & Whinston, 2014). Next, the discussion moves to the three geometric points mentioned above before presenting nine questions which allow students to provide estimates of the coefficients of the cubic cost function underlying a given cubic cost curve. Two final questions round out the analysis by introducing competitive market to the discussion. These questions are provided with answers here, but a supplementary Excel file provides the same questions without answers for ready distribution to students. This file also allows faculty to create new TC scenarios for homework or exams. To distribute to students, simply delete all sheets from the file except the BackgroundAndQuestions sheet. This sheet provides the TC graph that forms the basis for this analysis as well as the algebraic and geometric background necessary to determine the TC(Q) function that produced the TC graph.

The Algebra of Cubic Cost

Cost functions need not be cubic, but this form offers the simplest algebraic means of describing a TC or VC function that is first convex downward then convex upward. A typical short run cost curve exhibits this changing convexity due to changing marginal productivity of the variable factors of production. Initial increasing marginal productivity is eventually replaced by declining marginal productivity due to the law of diminishing returns. To obtain this shape we must restrict the cubic equation TC(Q) with restrictions on coefficients *a* – *d*:

$$\mathrm{TC(Q)} = a + b\mathrm{Q} + c\mathrm{Q}^2 + d\mathrm{Q}^3 \qquad \text{with } a, b, \text{ and } d > 0,\ c < 0 \text{ and } c^2 < 3bd\,. \qquad \textbf{(1)}$$

The point of inflection occurs when Q > 0 if a, b, and $d > 0$, and $c < 0$.$c^2 < 3bd$ is required to ensure that TC(Q) is an increasing function of Q.

This is a short run cost function because parameter $a > 0$ represents fixed cost.

$$\mathrm{FC(Q)} = a \qquad \text{given the total cost function in Equation 1.} \qquad \textbf{(2)}$$

Each of the other five short run cost functions (VC, ATC, AVC, AFC, MC) are obtained by algebraic manipulation of Equation 1. Since a is fixed cost, and TC = FC + VC, the rest of Equation 1 is variable cost.

$$VC(Q) = bQ + cQ^2 + dQ^3 \qquad \text{given the total cost function in Equation 1.} \qquad \textbf{(3)}$$

The short run average cost functions are obtained by dividing each of the above by Q.

$$ATC(Q) = a/Q + b + cQ + dQ^2 \qquad ATC(Q) = TC(Q)/Q = AFC(Q) + AVC(Q). \qquad \textbf{(4)}$$

$$AFC(Q) = a/Q \qquad AFC(Q) = FC/Q = ATC(Q) – AVC(Q). \qquad \textbf{(5)}$$

$$AVC(Q) = b + cQ + dQ^2 \qquad AVC(Q) = VC(Q)/Q = ATC(Q) – AFC(Q). \qquad \textbf{(6)}$$

Marginal cost is the slope of the TC and VC curves. Given TC in Equation 1 or VC in Equation 3, MC is given by,

$$MC(Q) = b + 2cQ + 3dQ^2 \qquad MC(Q) = \Delta TC/\Delta Q. \qquad \textbf{(7)}$$

Equation 7 describes the slope of TC and VC and can be found by taking the derivative of either TC or VC. If you have not seen calculus before, then this is simply a fact that can be used whenever you have a cubic cost function.

FACT: *You can obtain MC from a cubic cost function by applying Rules 1 and 2 below to the total cost function.*

Rule 1) Drop the fixed cost component a – this only shifts TC up or down, but does not change slope at a given output level.

Rule 2) The linear, quadratic, and cubic terms (b, c, and d) become the constant, linear and quadratic coefficients once they are multiplied by 1, 2 and 3.

The Geometry of Cubic Cost

The convex down then convex up shape of the TC function means that MC must initially decline prior to eventually increasing. This also means that AVC initially declines prior to eventually increasing. If TC is a cubic function, then both AVC and MC are quadratic functions. As such, both are symmetric about their minimum values. These minimums are readily determined given the equations for AVC and MC. The output level where AVC is minimized, Q_v, satisfies:

$$Q_v = -c/(2d) \qquad \text{given the AVC function in Equation 6.} \qquad \textbf{(8)}$$

The output level where MC is minimized, Q_m, satisfies:

$$Q_m = -c/(3d) \qquad \text{given the MC function in Equation 7.} \qquad \textbf{(9)}$$

A quick perusal of Equations 8 and 9 provides a useful rule for graphing cubic per-unit cost curves. The output level where MC is minimized is 2/3 the level where AVC is minimized:

$$Q_m = (2/3)\cdot Q_v\,. \qquad \textbf{(10)}$$

For example, if TC is cubic and AVC is minimized at output level $Q_v = 30$, then MC is minimized at output level $Q_m = 20$.

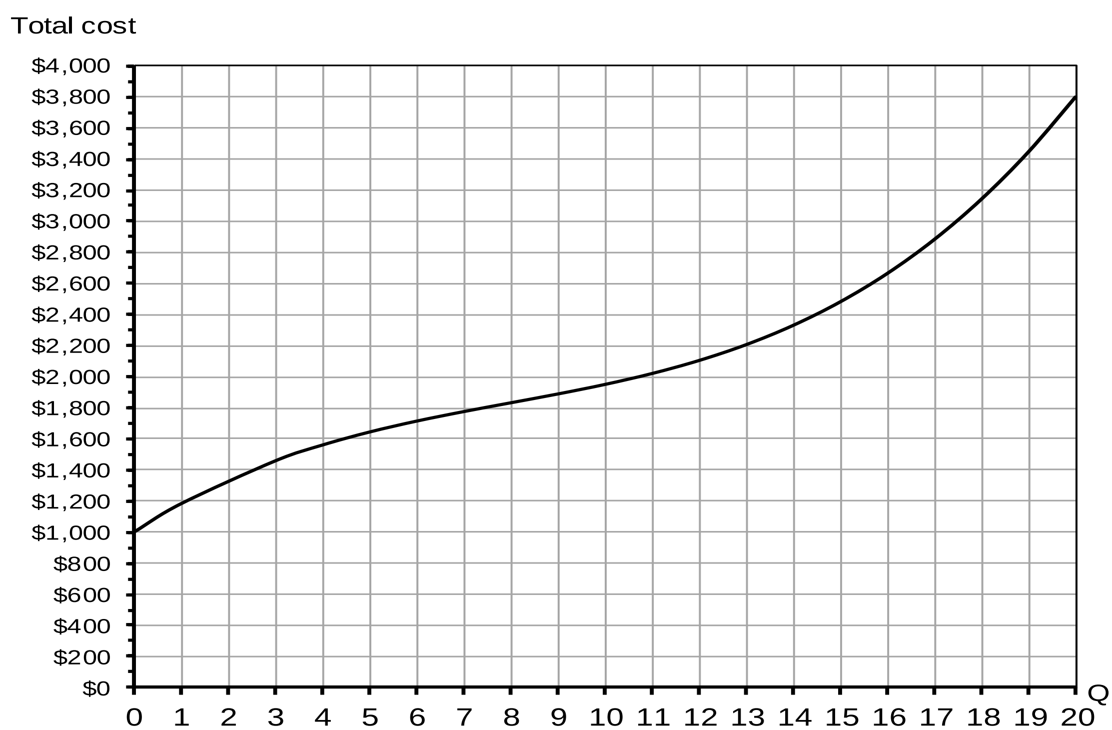
Marginal cost is the change in TC for a given change in output. The heuristic device presented in introductory microeconomics courses is to imagine a 1 unit change in output but the most accurate description of slope, especially when output is finely divisible, is the slope of the tangent to the curve rather than the slope of the secant to the curve. To find the value of MC at any output level, simply sketch the tangent at that output level and calculate the slope of that tangent using rise/run. Such calculations are more accurate, the longer the run (change in Q), as long as a straight-edge is used to draw the line.

An average value for a point on any type of total curve can be derived using a simple geometric trick. The slope of the chord connecting the origin to a point on that curve is the average value at that point. The slope of the chord connecting the origin to a point on TC is ATC at that point because ATC = TC/Q. In Figure 1, ATC(20) = $190 because the slope of the chord connecting (0, $0) and (20, $3,800) is $190 ($190 = [$3,800 – $0]/[20 – 0]).

If the TC curve is given but the VC curve is not, then it is easier to simply imagine that the TC intercept point on the vertical axis is a "new origin" than it is to draw the VC curve parallel to the TC curve. Considered relative to the new origin, the resulting curve is geometrically a VC curve. AVC is the slope of the chord connecting this new origin to points on the curve. In Figure 1 below, AVC(20) = $140 because the slope of the chord connecting the points (0, $1,000) and (20, $3,800) is $140 ($140 = [$3,800 – $1,000]/[20 – 0]).

Deriving TC(Q) and graphing per-unit cost curves given a cubic TC curve

Figure 1. A cubic total cost curve.



Questions 1 through 9 refer to the total cost curve depicted in Figure 1. It will help to use a pencil and straight-edge when answering Questions 1 through 8.

1. A) For what range of output is total cost convex upward?

B) What is true about the marginal product of labor over this range of output?

Answer 1.

A) Although it is not possible to determine exactly without knowing the underlying cost function, the point of inflection is about at Q = 8.

Therefore, the total cost function is convex upward for Q > 8.

B) The marginal product of labor declines over this range of output.

2. What is fixed cost given this short run production process?

Answer 2.

Fixed cost is the vertical axis intercept of the total cost function.

In this instance it is FC = $1,000.

3. A) What is the approximate output level at which marginal cost is minimized?

B) What is true about the marginal product of labor at this output level?

C) Provide an approximate dollar per-unit value to marginal cost at this output level.

Hint**:** This is easier to answer if you use a straight-edge to draw the appropriate tangent line.

Answer 3.

A) Marginal cost is the slope of the total cost curve. It is minimized at the point of inflection which in the above graph occurs at about Q = 8.

B) The marginal product of labor is maximized at the output level where marginal cost is minimized.

C) The dollar size of marginal cost is obtained by drawing a tangent at the point of inflection.

This is trickiest part of the graphical analysis connecting total cost and per-unit cost. In this instance the tangent has slope of approximately 56. This is the red line in Figure 3. Put another way, MC(8) = $56.

4. A) For what range of output does the average variable cost of production increase?

Hint**:** This is easy to see if you do not try to redraw the total cost curve but instead simply use the vertical intercept of the TC curve as a “new” axis from which to draw AVC chords.

B) What can be said about marginal cost over this range of output?

C) What is minimum AVC in this instance? At what output level does this occur?

Answer 4.

A) NOTE: By starting at the FC point, the TC curve effectively becomes a VC curve.

AVC is increasing past the point of minimum AVC. The way to find the minimum AVC point is to start at the FC point (0, \$1000) and rotate a ruler up until it just "kisses" the curve. This occurs at Q = 12. This is the blue "chord" in Figure 3.

Note that the rule described in Equation 10 holds, $Q_m = 2/3Q_v$ in this instance, $8 = (2/3)\cdot 12$.

For output greater than the point where the chord kisses the curve, we have increasing AVC. Therefore, Q > 12 has increasing AVC.

B) For Q > 12, MC > AVC. This is seen geometrically by noting that the slope of the tangent is greater than the slope of the "chord" for output levels greater than 12.

C) The slope of the Minimum AVC is approximately: AVC(12) = \$91.

(Do not be worried if you don't get exactly the same answer. This level of accuracy is not expected but is simply based on following the calculations through.)

5. A) For what range of output does average total cost of production decline?

B) What is minimum ATC in this instance? At what output level does this occur?

Answer 5.

A) ATC is declining up to the minimum sloping chord to the TC curve (using the origin as the pivot point).

B) Minimum ATC occurs at Q = 15. At this point the slope of the chord is 165. Put another way, ATC(15) = \$165. This is the green chord in the upper panel of Figure 3.

6. What is an approximate value of AVC(0) = MC(0)?

Answer 6.

The slope of TC at Q = 0 is MC(0) and AVC(0). This is approximately \$200. This is the gold chord in Figure 3.

7. If you were graphing ATC(Q) and AVC(Q), what is ATC(Q) – AVC(Q) at output levels Q = 8, 10, 12.5 and 20?

Answer 7.

This difference is AFC(Q) as can be seen from Equation 5.

Since AFC(Q) = FC/Q, we can easily derive these values.

ATC(8) – AVC(8) = AFC(8) = 1000/8 = 125.

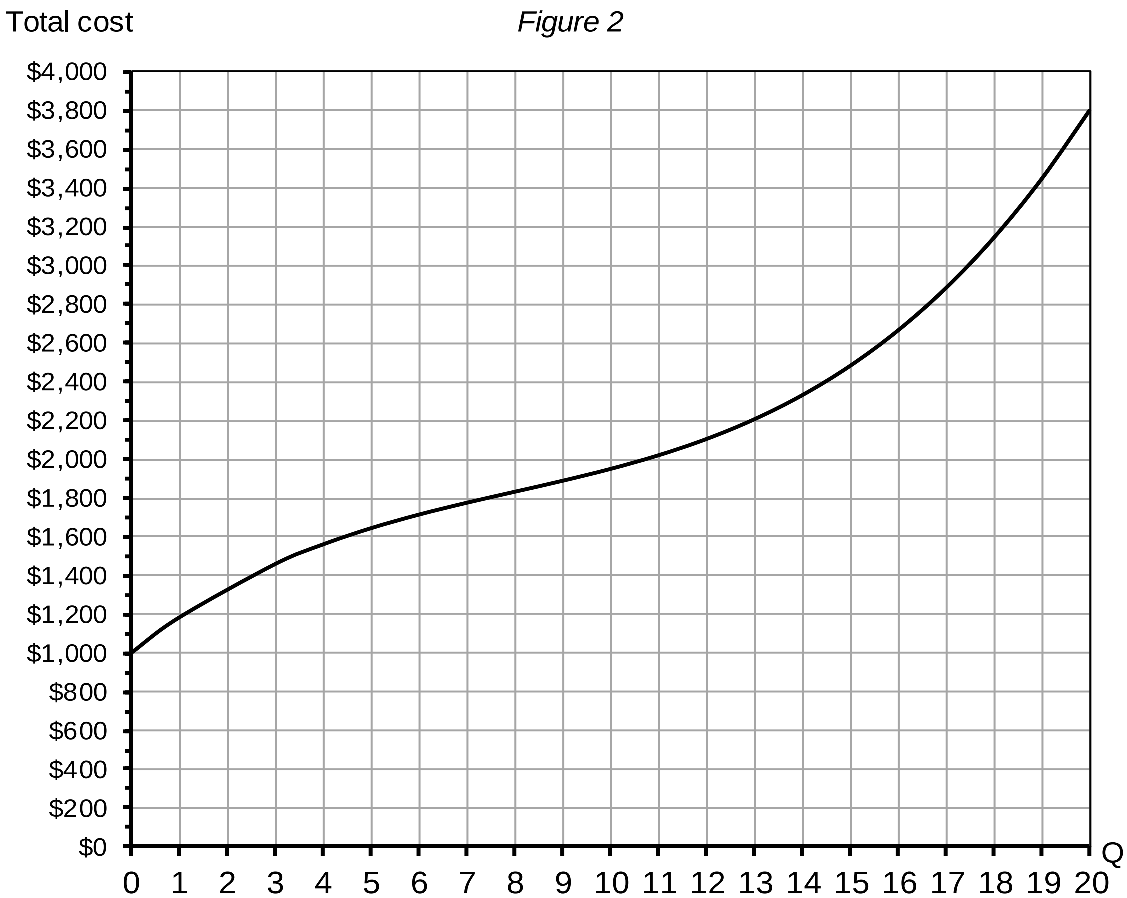
ATC(10) – AVC(10) = AFC(10) = 1000/10 = 100.

ATC(12.5) – AVC(12.5) = AFC(12.5) = 1000/12.5 = 80.

ATC(20) – AVC(20) = AFC(20) = 1000/20 = 50.

8. Use your answers to Questions 1 through 7 to graph per-unit cost curves. Make sure to provide an appropriate scale for the vertical axis in the per-unit cost panel of Figure 2.

Figure 2


Total cost
$4,000
$3,800
$3,600
$3,400
$3,200
$3,000
$2,800
$2,600
$2,400
$2,200
$2,000
$1,800
$1,600
$1,400
$1,200
$1,000
$800
$600
$400
$200
$0
Q
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

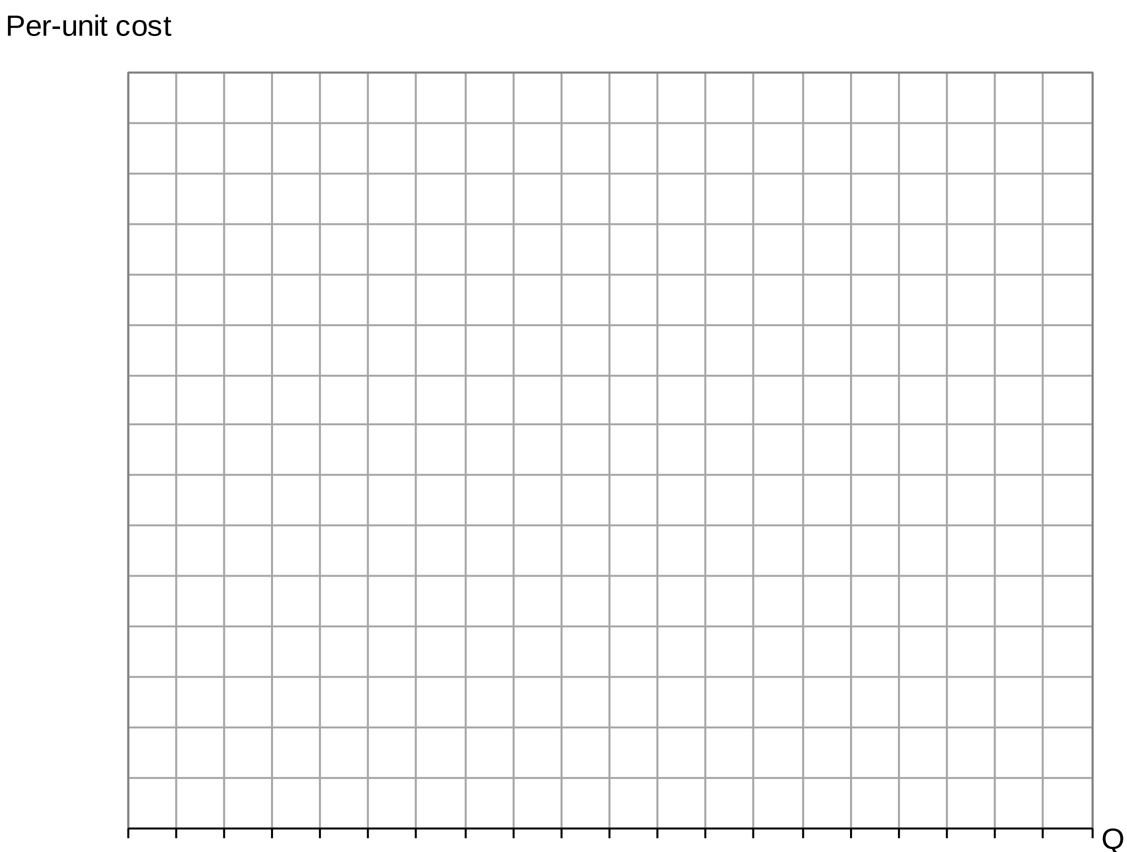
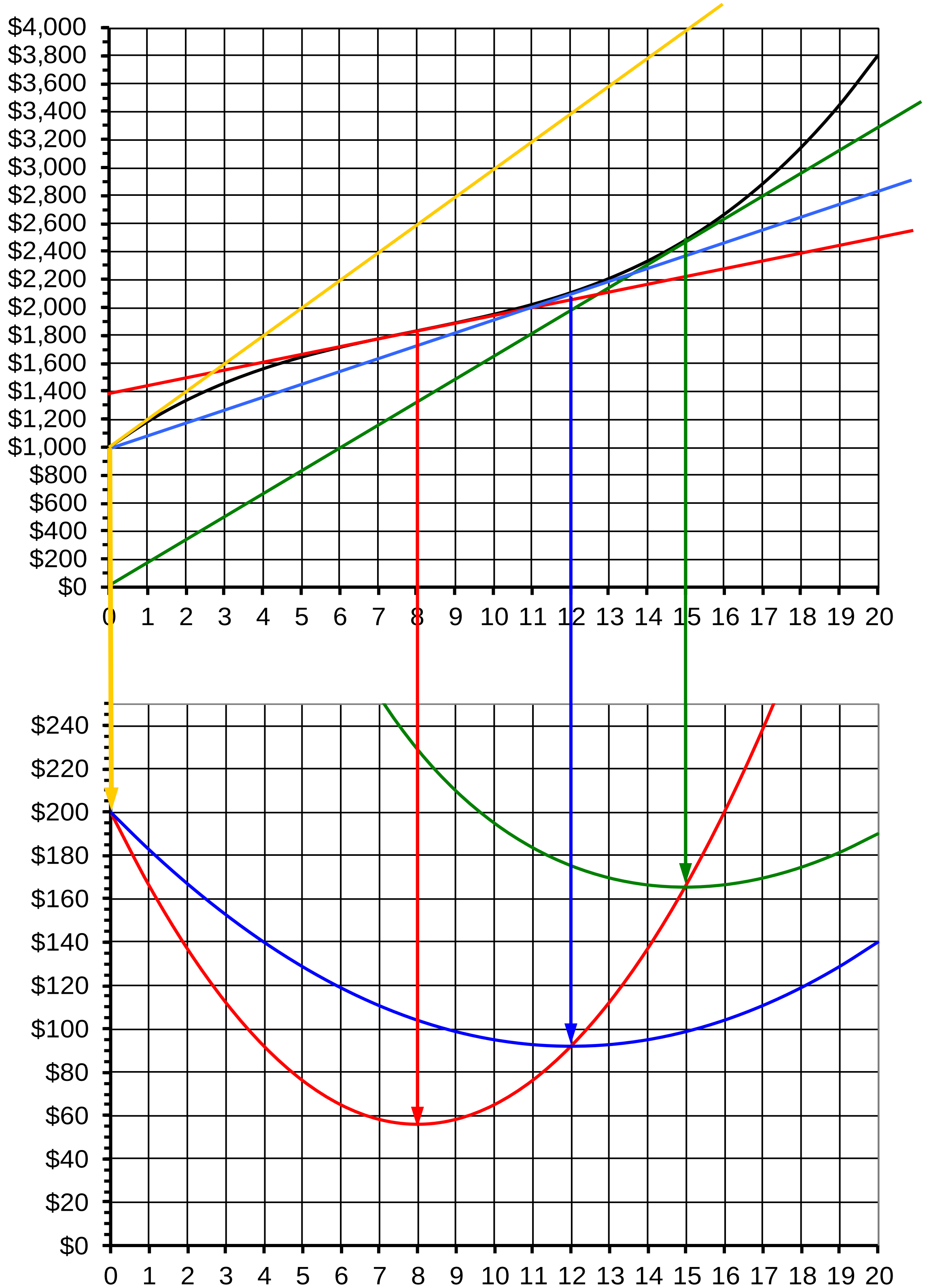

Per-unit cost
Q
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Figure 3. Answer Graph:
Marked up TC linked to per-unit cost curves.

Tangency Q	Tangency $/Q (calculated as rise/run)
15	**ATC min @ = (3300-0)/20 = 165.**
12	**AVC min @ = (2820-1000)/20 = 91.**
8	**MC min @ = (2500-1380)/20 = 56.**
0	**MC(0) = AVC(0) = (4000-1000)/15 = 200.**



9. Suppose the total cost curve depicted in Figure 1 is based on the cubic functional form: $TC(Q) = a + bQ + cQ^2 + dQ^3$. What are approximate values for the coefficients *a*, *b*, *c*, and *d*?

***Hints*:** This can be accomplished by answering a couple of questions. 1) What is fixed cost? 2) What is MC(0)? 3) What can we say about the relation between *c* and *d* at minimum AVC? 4) Use this information to evaluate TC(Q) at an output level such as Q = 10.

NOTE: Each individual's answers may differ from one another but they will be close to one another as long as slope estimates are approximately correct.

Answer 9.

Two coefficients are quite easy to obtain: FC = a and MC(0) = b. You derived above (in Questions 2 and 6) that these values are ***a = 1000*** *and* ***b = 200.***

Minimum AVC occurs when Q = -c/(2d) according to Equation 8. Since minimum AVC occurs at Q = 12 (Question 4), we obtain: c = -24d.

At Q = 10 we have TC(10) = 1950

Based on this we have: $1950 = 1000 + 200\cdot10 - 24d\cdot100 + d\cdot1000$

Regrouping and simplifying we obtain: $1400d = 1050$, *or* ***d = 0.75****.*

Based on c = -24d we obtain: ***c = -18.***

Therefore the cubic TC equation is: $\mathbf{TC(Q) = 1000 + 200Q - 18Q^2 + 0.75Q^3}$.

10. Imagine you are embedded in a competitive market given the above cost information.

A) Suppose P = \$200, how many units should you produce and what profits result?

B) Suppose P = \$100, how many units should you produce and what profits result?

C) At what price will you decide to shut down rather than produce along your MC curve?

Answer 10.

A) *Profits are positive because P > minATC. This is easiest to see in the TC figure but can also be seen using the per-unit cost figure. Q in this instance is 16 and occurs where the slope of TC (i.e. MC) equals the slope of TR (note that MC(16) = 200), and π = TR(16) – TC(16) = 3200 – 2660 = \$540 profits result.*

This scenario is depicted by the green information in Figure 4.

Note also that MC(16) = 200 is consistent with a symmetric MC curve given minimum MC occurs at Q = 8 and MC(0) = 200.

B) *Profits are negative because P < minATC. This is easiest to see in the TC figure but can also be seen using the per-unit cost figure. Q in this instance is 12.4 where the slope of TC = 100 (note that MC(12.4) = 100).*

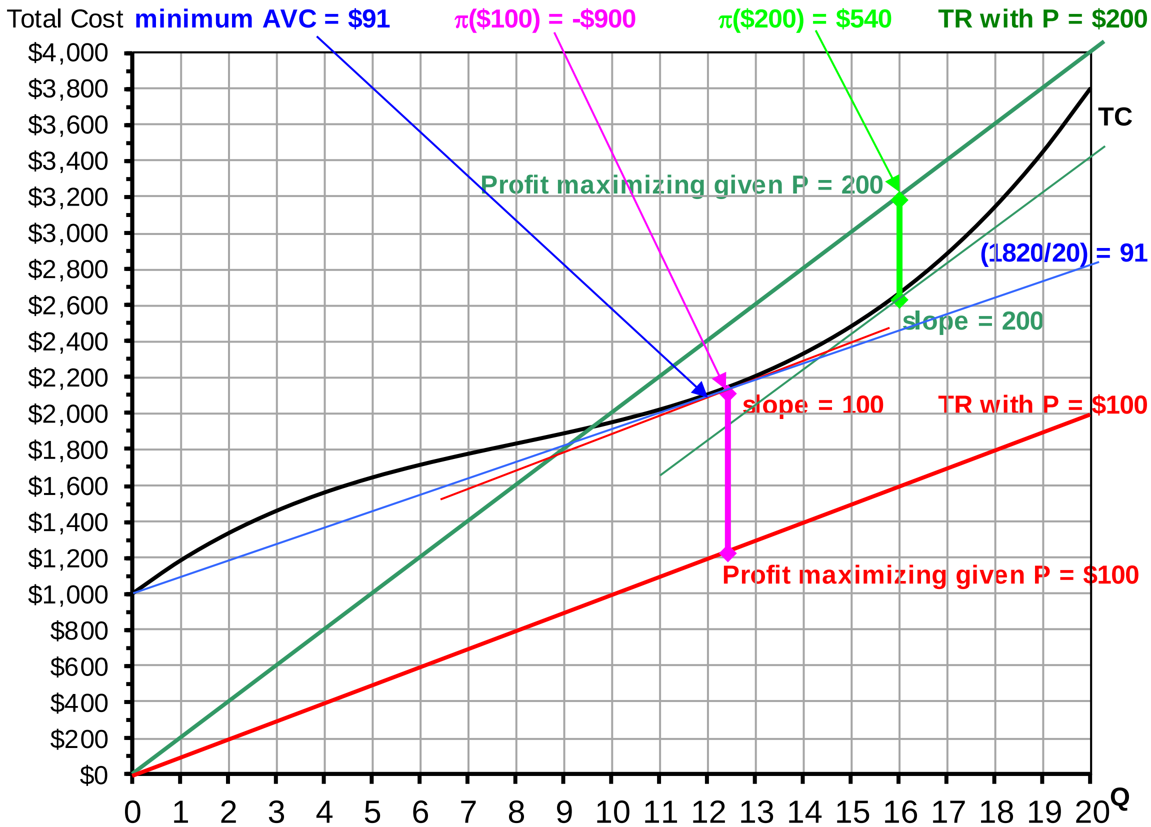
π *= TR(12.4) – TC(12.4)= 1240 – 2140 = -900 results.*

This is better than shutting down because losing $900 is better than losing $1,000 in the short run. This scenario is depicted by the red/pink information in Figure 4.

C) The shut-down point occurs if price equals minimum AVC $/Q level. This was calculated in Question 4 to be $91 at an output level of 12.

This scenario is depicted by the blue information in Figure 4.

Figure 4. Answer Graph for Question 10, Short Run Competitive Analysis.



11. What happens in the competitive market described in Question 10 as you move from the short run to the long run? Provide separate answers based on scenario A and B. In both events, what is the price in the long run in this market if all firms in the market face similar cost curves?

Answer 11.

In scenario A, positive profits signal entry. Supply increases and price declines until it achieves its long run equilibrium of P_{LR} = $165 the value of minimum ATC in Question 5.

As the price declines to $165, the green TR curve from Figure 4 rotates down until it is just tangent to TC at Q = 15.

In scenario B, negative profits signal exit. Supply decreases and price increases until it achieves its long run equilibrium of P_{LR} = $165 the value of minimum ATC in Question 5.

As the price increases to \$165, the red TR curve from Figure 4 rotates up until it is just tangent to TC at Q = 15.

In both instances, the resulting TR line will look just like the green minimum ATC line in the upper panel of Figure 3 that is just tangent to TC at Q = 15.

In the lower panel of Figure 3 this occurs at (15, \$165), the intersection of MC and ATC.

This managerial economics exam question formed the basis for the above analysis.

(26 points total) Suppose you have estimated a total cost function such as the one that is graphed in Figure 2.

A;13) On the graph beneath the one provided, sketch your best guess regarding what ATC, AVC and MC look like. Pay particular attention to the point of minimum MC, ATC, and AVC. Given the units which are presented on the top graph, estimate appropriate units for the bottom graph (tick marks on the vertical axis have been provided; you must add numbers – make sure to use the entire graph).

B;4) Suppose this total cost function is the cubic function: $C = a + b*Q + c*Q^2 + d*Q^3$. Provide numerical values for *a, b, c* and *d* based on your above analysis.

C;6) If the price of the product is \$200/unit, about how many units of the good would you suggest this firm produce? Explain, being sure to estimate SR profits or losses that result from your production decision.

D;3) What happens as we move from the short to the long run both in terms of entry/exit and what is the LR equilibrium price in this market?

***NOTE*:** You can accomplish C&D without attempting part B so don't get bogged down on B.

Grading Rubric for Exam Question

<u>A) 6 on general shapes</u>

1 general shape of MC

1 general shape of AVC

1 AVC = MC at Q = 0

1 AVC = MC at $Q = Q_{minAVC}$

1 general shape of ATC

1 ATC = MC at $Q = Q_{minATC}$

<u>3 on Q values</u>

1 min AVC occurring at Q of minimum supporting line (drop down) @ Q = 12 +-1

1 min MC occurring at point of inflection @ Q = 8 +-1

1 min ATC occurring at Q of minimum supporting line (drop down) @ Q = 15 +-1

4 on \$/Q values

1 *numerically appropriate \$/Q at $Q = Q_{minAVC}$* *\$91+-5*

1 *numerically appropriate \$/Q at Q = 0* *\$200+-20*

1 *numerically appropriate \$/Q at $Q = Q_{minMC}$* *\$56 +- 10*

1 *numerically appropriate \$/Q at $Q = Q_{minATC}$* *\$165 +- 5*

B) 4 on equation

1 *a must be close to 1000*

1 *b must equal the \$/Q value given at Q=0 (answer should be close to b = 200)*

1 *c can be calculated from the \$/Q and Q values (answer should be close to c = -18)*

1 *d can be calculated from the \$/Q and Q values (answer should be close to d = 0.75)*

C) 3 on finding Q(\$100)

1 *if Price = \$200, produce along the marginal cost curve*

1 *as long as price is above minAVC*

1 *so produce Q=16+-1*

3 on finding SRprofits

1 *profits = TR-TC*

1 *TR is 200Q and TC is obtained from TC function*

1 *Profits = 540 +-20*

D) 3 on finding LRequilibrium

1 *In LR entry will occur*

1 *LR price will be minATC*

1 *numerical, same value as above minATC answer.*

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